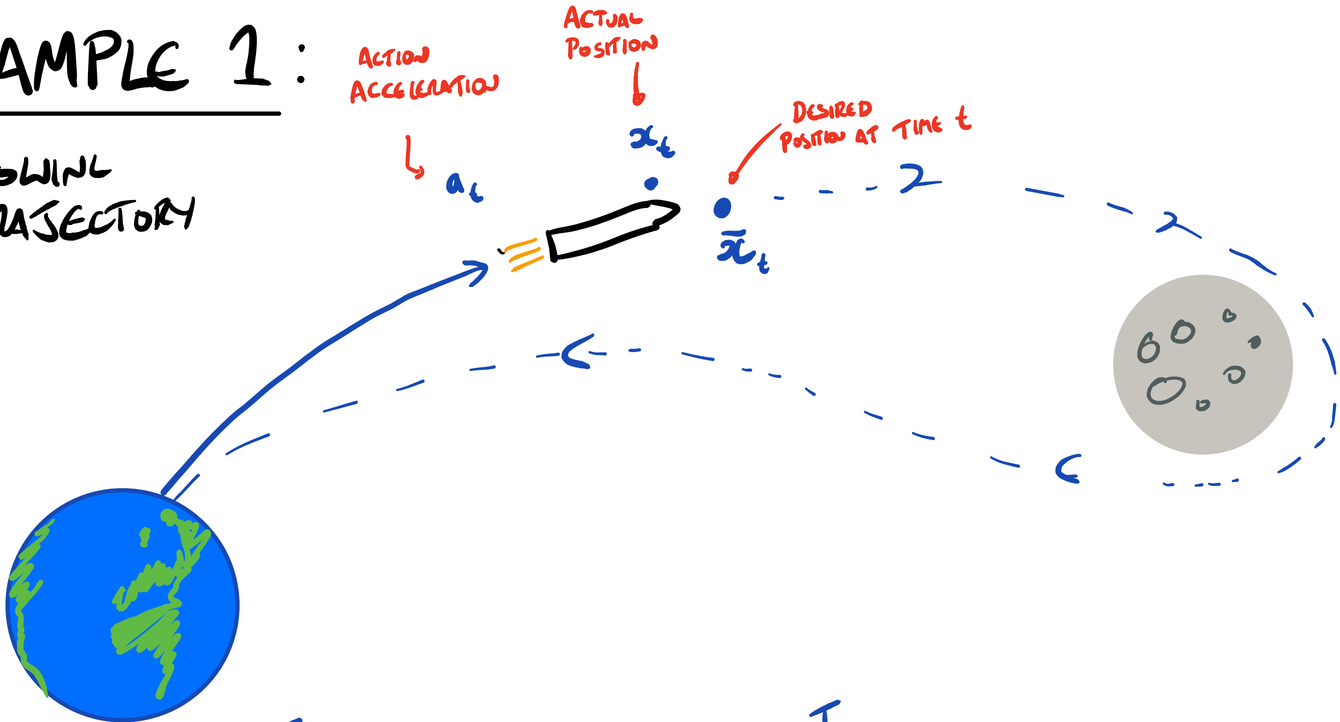
7. LINEAR-QUADRATIC REGULARIZATION & KALMAN FILTER

EXAMPLE 1:

FOLLOWING A TRAJECTORY



$$\text{MIN} \quad \sum_{t=1}^{T} \| x_t - \bar{x}_t \|^2 + \sum_{t=1}^{T} c \, \| a_t \|^2$$

$$\text{s.t.} \quad x_{t+1} = x_t + v_t$$

$$v_{t+1} = v_t + a_t$$

EXAMPLE 2:

TRADING WITH TRANSACTION COSTS

COST OF ASSET

COST OF TRANSACTION

VOLATILITY

CASH OUT

COST OF TRANSACTION

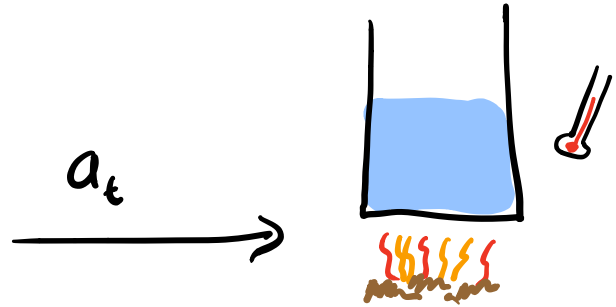
$$\min \mathbb{E}\left[\left[\sum_{t=1}^{T-1} a_t + k\, a_t^2 + (x_t + a_t)\, \Sigma_t\, (x_t + a_t) \right] - x_T + k\, x_T^2 \right]$$

$$\text{s.t.} \quad x_{t+1} = [x_t + a_t]\, \varepsilon_t$$

BECAUSE MULTIPLIED IS NOT QUITE LQR. BUT STILL FITS.

EXAMPLE 3:

HEATING AT THE CORRECT TEMPERATURE.



INFINITE TIME

TARGET STATE

$$\min \mathbb{E}\left[\sum_{t=1}^{\infty} \beta \, \| x_t - x^* \|^2 \right]$$

HEAT GOING IN

HEAT GOING OUT

NOISE

s.t.

$$x_t = x_{t-1} + a_t - c + \varepsilon_t$$

$$y_t = x_{t-1} + \eta_t$$

NOISE IN OBSERVATION.

LINEAR QUADRATIC REGULARIZATION (LQR)

CONSIDER

QUADRATIC OBJECTIVE

MINIMIZE $\sum_{t=0}^{T-1} \{ x_t^T R x_t + a_t^T Q a_t \} + x_T^T R x_T$

LINEAR MOVEMENT

SUBJECT TO $x_t = A x_{t-1} + B a_{t-1}, \quad t = 1, \dots, T$

OVER $a_0, \dots, a_{T-1} \in \mathbb{R}^m$

$x^T R x \geq 0$

A & B ARE MATRICES

R & Q ARE POSITIVE SEMI-DEFINATE MATRICES.

SOLVING LQR

THEOREM: THE VALUE FUNCTION IS

$$V_t(x) = x^T \Lambda_t x$$

← TIME SO FAR

WHERE Λ_t SATISFIES

THE RICCARTI EQUATION.

$$\Lambda_t = R + A^T \Lambda_{t+1} A - A^T \Lambda_{t+1} B (Q - B^T \Lambda_{t+1} B)^{-1} B^T \Lambda_{t+1} A$$

THE OPTIMAL ACTION IS

$$a_t^* = -\underbrace{(Q + B^T \Lambda_t B)^{-1} B^T \Lambda_t}_{= G_t} A x$$

PROOF: BELLMAN EQUATION IS

$$L_{t-1}(x) = \min_a \{ x^T R x + a^T Q a + L_t(Ax + Ba) \}$$

[CLEARLY $L_T(x) = x^T R x$
INDUCTION]

ASSUME $L_t(x) = x^T \Lambda_t x$

SYMETRIC

GIVES

$$\min_a \; x^T R x + a^T Q a + x^T A^T \Lambda_t A x + 2 a^T B^T \Lambda_t A x + a^T B^T \Lambda_t B a$$

DIFFERENTIATE

$$0 = \nabla_a \{\cdot\} = 2Qa + 2B^T \Lambda_t A x + 2B^T \Lambda_t B a$$

PROOF (CONT): IMPLIES

$$a_* = -\left[Q + B^T \Lambda_t B\right]^{-1} B^T \Lambda_t A x$$

SUBSTITUTE IN BELLMAN EQN:

$$x^T \Lambda_{t-1} x = x^T R x + a_*^T Q a_* + x^T A^T \Lambda_t A x \underbrace{+ 2 a_*^T B^T \Lambda_t A x + a_*^T B^T \Lambda_t B a_*}_{= -x^T A^T \Lambda_t B [Q + B^T \Lambda_t B]^{-1} B^T \Lambda_t A x}$$

[CHECK YOURSELF]

$\therefore$ REQUIRE $\forall x$

$$x^T \Lambda_t x = x^T \left\{ R + A^T \Lambda_{t+1} A - A^T \Lambda_{t+1} B \left(Q - B^T \Lambda_{t+1} B\right)^{-1} B^T \Lambda_{t+1} A \right\} x$$

$\square$.

LQR INCLUDING "NOISE"

MINIMIZE $\sum_{t=0}^{T-1} x_t^T R x_t + a_t^T Q a_t + x_T^T R x_T$

SAME OBJECTIVE

SUBJECT TO $x_t = A x_{t-1} + B a_{t-1} + \underline{\varepsilon_t}$

R.V. NOISE

WHERE $\mathbb{E}[\varepsilon_t] = 0$

$\mathbb{E}[\varepsilon_t \varepsilon_t^T] = N$

SOLVING LQR WITH NOISE

THEOREM: THE VALUE FUNCTION IS

$$V_t(x) = x^T \Lambda_t x + \gamma_t$$

WHERE Λ_t & a^* ARE IDENTICAL TO LQR

[FROM BEFORE WITHOUT NOISE]

PROOF: BELLMAN EQUATION IS

$$L_{t-1}(x) = \min_a \{ x^T R x + a^T Q a + \mathbb{E}[L_t(Ax + Ba + \underline{\varepsilon})] \}$$

ASSUME $L_t(x) = x^T \Lambda_t x + \underline{\gamma_t}$

GIVES

$$x^T \Lambda_{t-1} x + \gamma_{t-1} = \min_a \underbrace{x^T R x + a^T Q a + x^T A^T \Lambda_t A x + 2 a^T B^T \Lambda_t A x + a^T B^T \Lambda_t B a}_{\text{SAME AS BEFORE}}$$

$$+ \underbrace{\mathbb{E}[\varepsilon^T \Lambda_t \varepsilon]}_{= \sum_{ij} \Lambda_{t,ij} \mathbb{E}[\varepsilon_i \varepsilon_j] = tr(\Lambda_t N)} + \underline{\gamma_t}$$

SO EQUATING TERMS

- Λ_{t-1} SATISFIES RICCARTI RECURSION
- $\gamma_{t-1} = tr(\Lambda_t N) + \gamma_t$ $\square$

LQR WITH NOISE & IMPERFECT OBSERVATION

MINIMIZE $\sum_{t=0}^{T-1} x_t^T R x_t + a_t^T Q a_t + x_T^T R x_T$

SUBJECT TO $x_t = A x_{t-1} + B a_{t-1} + \varepsilon_t$

$$y_t = C x_t + \delta_t$$

YOU DON'T OBSERVE x_t ← [OR ε_t OR δ_t]

YOU OBSERVE y_t ← [A NOISY LINEAR FUNCTION OF x_{t-1}]

SO YOU USE INFORMATION

$$F_t = (y_t, \ldots, y_1, a_{t-1}, \ldots, a_0)$$

SOLVING WITH NOISE & IMPERFECT OBSERVATION

THEOREM: THE OPTIMAL VALUE FUNCTION IS

$$V_t(F_t) = \mathbb{E}\left[x_t^T \Lambda_t x_t \mid F_t\right] + \sum_{\tau=t}^{T-1} \mathbb{E}\left[\Delta_t^T \tilde{\Lambda}_\tau \Delta_t \mid F_t\right] + \gamma_t$$

& THE OPTIMAL CONTROL IS $a_t^* = K_t \bar{x}_t$

HERE K_t, Λ_t & γ_t ARE THE SAME AS BEFORE.

$$\bar{x}_t = \mathbb{E}\left[x_t \mid F_t\right] \quad \& \quad \Delta_t = x_t - \bar{x}_t$$

$$\tilde{\Lambda}_t = \underbrace{R + A^T \Lambda_{t+1} A - \Lambda_t}_{\text{the SEMI-DEFINATE.}}$$

CERTAINTY EQUIVALENCE:

[WE PROVE THE THEOREM IN A COUPLE OF MINUTES]

THE LAST RESULT IS INTERESTING BECAUSE

$$a_t = G_t \bar{x}_t$$

CONTROL SAME AS FOR FULL INFORMATION

JUST USE YOUR BEST ESTIMATE OF x_t i.e. $\bar{x}_t = \mathbb{E}[x_t | F_t]$

THIS IS CALLED: CERTAINTY EQUIVALENCE

i.e. YOU TREAT THE MEAN AS IF IT WAS CERTAIN.

[YOU STILL NEED TO BE ABLE TO CALCULATE $\bar{x}_t$
WE DO THIS SHORTLY WITH THE KALMAN FILTER].

CERTAINTY EQUIVALENE: [SO WHAT MAKES CERTAINTY EQUIVALENCE HOLD]

THE TERM $\sum_{\tau=t}^{T-1} \mathbb{E}[\Delta_\tau^T \tilde{\Lambda}_\tau \Delta_\tau \mid F_t]$ IN $V_t(F_t)$ DOESN'T LOOK GOOD!

$\Delta_\tau \doteq x_\tau - \bar{x}_\tau$ IF a_t CHANGES THEN IT LOOKS LIKE Δ_τ CHANGES IN A WAY THAT DEPENDS ON INTERMEDIATE ACTIONS $a_{t+1}, \dots, a_{\tau-1}$

MAYBE FINE BUT NOT GOOD FOR NEAT CLOSED FORM SOLUTIONS!

HOWEVER,

LEMMA: Δ_τ IS CONSTANT W.R.T. $a_0, \dots, a_{\tau-1}$; x_0.

PROOF:

$$x_\tau = A x_{\tau-1} + B a_{\tau-1} + \varepsilon_{\tau-1}$$

$$= A[A x_{\tau-2} + B a_{\tau-2} + \varepsilon_{\tau-2}] + B a_{\tau-1} + \varepsilon_{\tau-1}$$

$$= A^2 x_{\tau-2} + AB a_{\tau-2} + B u_{\tau-1} + A\varepsilon_{\tau-2} + \varepsilon_{\tau-1}$$

$$\vdots$$

PROOF: (CONT.)

$$x_\tau = A^\tau x_0 + \sum_{t=0}^{\tau-1} A^{\tau-1-t} B a_t + \sum_{t=0}^{\tau-1} A^{\tau-1-t} \varepsilon_t$$

$$\bar{x}_\tau = A^\tau x_0 + \sum_{t=0}^{\tau-1} A^{\tau-1-t} B a_t + \mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t} \varepsilon_t \,\middle|\, F_\tau\right]$$

$$\therefore\ x - \bar{x}_\tau = \sum_{t=0}^{\tau-1} A^{\tau-1-t} \varepsilon_t - \underbrace{\mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t} \varepsilon_t \,\middle|\, F_\tau\right]}_{F_\tau = (y_\tau, \ldots, y_1,\ a_{\tau-1}, \ldots, a_0)}$$

[SO IN PRINCIPLE DEPENDS ON a]

NOTICE

$$\begin{aligned} y_\tau &= C x_\tau + \delta_\tau \\ &= C A^\tau x_0 + \sum_{t=0}^{\tau-1} C A^{\tau-1-t} B a_t + \sum_{t=0}^{\tau-1} C A^{\tau-1-t} \varepsilon_t + \delta_\tau \end{aligned}$$

ALSO LET

$$y^\circ_\tau = C A^\tau x_0 + \sum_{t=0}^{\tau-1} C A^{\tau-1-t} \varepsilon_t + \delta_\tau$$

REMOVED THE ACTIONS

ANY POLICY $a_t = \pi(F_t) = \pi(y_{[0:t]}, a_{[0:t-1]})$

[DETERMINISTIC FN OR INDEPENDENT RV]

$a_{[0:t]} \doteq (a_0, \dots, a_t)$

HOWEVER, EQUALLY WE COULD TAKE

$$a_t = \pi^\circ(y^\circ_{[0:t]}, a_{[0:t-1]})$$

[AGAIN DETERMINISTIC]

I.e. CONDITIONING ON $F_\tau = (y_{[0:\tau]}, a_{[0:\tau-1]})$ IS THE SAME AS CONDITIONING ON $(y^\circ_{[0:\tau]}, \pi^\circ)$

SO

$$\mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t}\varepsilon_t \,\middle|\, F_\tau\right] = \mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t}\varepsilon_t \,\middle|\, y^\circ_{[0:\tau]}, \pi^\circ\right] = \underbrace{\mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t}\varepsilon_t \,\middle|\, y^\circ_{[0:\tau]}\right]}_{\text{DOESN'T DEPEND ON } a_{[0:t-1]}\text{!}}$$

$\therefore$

$$\Delta_\tau = x - \bar{x}_\tau = \sum_{t=0}^{\tau-1} A^{\tau-1-t}\varepsilon_t - \mathbb{E}\left[\sum_{t=0}^{\tau-1} A^{\tau-1-t}\varepsilon_t \,\middle|\, y^\circ_{[0:\tau]}\right]$$

DOES NOT DEPEND ON ACTIONS $\square$

[NOTICE THE PROOF WORKS BECAUSE x_τ IS A LINEAR FN OF $x_{\tau-1}, a_{\tau-1}, \varepsilon_{\tau-1}$]

PROOF OF THEOREM: THE RESULT IS TRUE AT TIME T

LET'S WORK BACKWARDS BY INDUCTION.

$$V_t(F_t) = \min_{a_t} \mathbb{E}\left[x_t^T R x_t + a_t^T Q a_t + V_{t+1}(F_{t+1}) \,\middle|\, F_t\right]$$

$$= \min_a \mathbb{E}\left[\underbrace{x_t^T R x_t} + a^T Q a + \underbrace{\mathbb{E}\left[x_{t+1}^T \Lambda_{t+1} x_{t+1} \mid F_{t+1}\right]} + \underbrace{\sum_{\tau=t+1}^{T-1} \mathbb{E}\left[\Delta_\tau^T \tilde{\Lambda}_\tau \Delta_\tau \mid F_{t+1}\right]} + \gamma_{t+1} \,\middle|\, F_t\right]$$

$$= \mathbb{E}\left[(\bar{x}_t + \Delta_t)^T R (\bar{x}_t + \Delta_t) \mid F_t\right]$$

$$= \bar{x}_t^T R \bar{x}_t + \mathbb{E}\left[\Delta_t^T R \Delta_t \mid F_t\right]$$

DOESN'T DEPEND ON a

$$= \mathbb{E}\left[x_{t+1}^T \Lambda_{t+1} x_{t+1} \mid F_t\right]$$

$$= \mathbb{E}\left[(Ax_t + Ba + \varepsilon_t)^T \Lambda_{t+1} (Ax_t + Ba + \varepsilon_t) \mid F_t\right]$$

$$= \mathbb{E}\left[x_t^T A^T \Lambda_{t+1} A x_t \mid F_t\right] + 2\bar{x}_t^T A^T \Lambda_{t+1} B a + a^T B^T \Lambda_{t+1} B a + tr(N\Lambda_{t+1})$$

$$= \bar{x}_t^T A^T \Lambda_{t+1} A \bar{x}_t + \mathbb{E}\left[\Delta_t^T A^T \Lambda_{t+1} A \Delta_t \mid F_t\right] + 2\bar{x}_t^T A^T \Lambda_{t+1} B a + a^T B^T \Lambda_{t+1} B a + tr(N\Lambda_{t+1})$$

CALL THIS I_{t+1} ["I" FOR IGNORE]

IT DOESN'T DEPEND ON a. BY LEMMA

SO CERTAINTY EQUIVALENCE HOLDS

THIS IS EXACTLY THE SAME AS LQR [WITH $x = \bar{x}_t$].

$$a^* = K_t \bar{x}_t$$

$$= \min_a \left\{ \bar{x}_t^T R \bar{x}_t + a^T Q a + \bar{x}_t^T A^T \Lambda_{t+1} A \bar{x}_t + 2 a^T B^T \Lambda_{t+1} A \bar{x}_t + a^T B^T \Lambda_{t+1} B a \right\}$$

$$+ \mathbb{E}\left[\Delta_t^T [R + A^T \Lambda_{t+1} A] \Delta_t \mid F_t\right] + I_{t+1} + \underbrace{tr(N\Lambda_{t+1}) + \gamma_t}_{\gamma_{t+1}}$$

$$\Rightarrow \bar{x}_t \Lambda_t \bar{x}_t$$

$$= \mathbb{E}\left[x_t \Lambda_t x_t \mid F_t\right] - \mathbb{E}\left[\Delta_t \Lambda_t \Delta_t \mid F_t\right]$$

PROOF (CONT.)

$$= \mathbb{E}\left[x_t^T \Lambda_t x_t \mid F_t\right] + \underbrace{\mathbb{E}\left[\Delta_t^T \underbrace{\left[R + A^T \Lambda_{t+1} A^T - \Lambda_t\right]}_{\tilde{\Lambda}_t} \Delta_t \mid F_t\right] + I_{t+1}}_{I_t} + \gamma_{t+1}$$

$\square$

THE KALMAN FILTER

SO FOR LQG PROBLEMS WE NEEDED TO KNOW

$$\bar{x}_t = \mathbb{E}[x_t \mid F_t]$$

WHERE

$$x_t = A x_{t-1} + B a_{t-1} + \varepsilon_{t-1}$$

$$y_t = C x_t + \delta_t$$

ASSUME GAUSSIAN

$$\varepsilon_t \sim N(0, \Sigma^{\varepsilon}) , \quad \delta_t \sim N(0, \Sigma^{\delta})$$

THE KALMAN FILTER

The Kalman filter has two update stages: a prediction update and a measurement update. These are

$$\bar{\boldsymbol{x}}_{t+1\,|\,t} = A\bar{\boldsymbol{x}}_{t+1\,|\,t} + Ba_t \qquad \text{(Predict-1)}$$
$$P_{t+1\,|\,t} = AP_{t\,|\,t}A^\top + \Sigma_t^\epsilon \qquad \text{(Predict-2)}$$

and

$$\bar{\boldsymbol{x}}_{t+1} = \bar{\boldsymbol{x}}_{t+1\,|\,t} + K_t(\boldsymbol{y}_{t+1} - C\bar{\boldsymbol{x}}_{t+1\,|\,t}) \qquad \text{(Measure-1)}$$
$$P_{t+1} = P_{t+1\,|\,t} - K_t C P_{t+1\,|\,t} \qquad \text{(Measure-2)}$$

where

$$K_t = P_{t+1\,|\,t}C^\top(CP_{t+1|t}C^\top + \Sigma_t^\delta)\,.$$

The matrix K_t is often referred to as the Kalman Gain. Assuming the initial state $\boldsymbol{x}_0$ is known and deterministic $P_{0|0} = 0$ in the above/

A RESULT ABOUT NORMAL R.V.s

Prop 25. *Let u be normally distributed vector with mean $\bar{u}$ and covariance Σ_u, i.e.*

$$\boldsymbol{u} \sim \mathcal{N}(\bar{\boldsymbol{u}}, \Sigma_{\boldsymbol{u}}).$$

i) For any matrix A and (constant) vector $\mathbf{c}$, we have that

$$A\boldsymbol{u} + \boldsymbol{c} \sim \mathcal{N}(A\bar{\boldsymbol{u}} + \boldsymbol{c}, A\Sigma_{\boldsymbol{u}}A^\top).$$

ii) If we take $\boldsymbol{u} = (\boldsymbol{v}, \boldsymbol{w})$ then $\boldsymbol{w}$ conditional on $\boldsymbol{v}$ gives

$$(\boldsymbol{w}|\boldsymbol{v}) \sim \mathcal{N}(\bar{\boldsymbol{w}} + \Sigma_{wv}\Sigma_{vv}^{-1}(\boldsymbol{v} - \bar{\boldsymbol{v}}), \Sigma_{ww} - \Sigma_{wv}\Sigma_{vv}^{-1}\Sigma_{vw})$$

iii) $Var(A\boldsymbol{u}) = A\Sigma_u A^\top$, $Cov(A\boldsymbol{u}, B\boldsymbol{u}) = A\Sigma_u B^\top$.

<u>PROOF</u>: MGFs $\square$.

MAIN RESULT ON KALMAN FILTER

Thrm 26.

$$[\boldsymbol{x}_{t+1}|\boldsymbol{y}_{[0:t]}, \boldsymbol{a}_{[0:t]}] \sim \mathcal{N}(\bar{\boldsymbol{x}}_{t+1|t}, P_{t+1|t})$$
$$[\boldsymbol{x}_{t+1}|\underbrace{\boldsymbol{y}_{[0:t+1]}, \boldsymbol{a}_{[0:t]}}_{F_{t+1}}] \sim \mathcal{N}(\bar{\boldsymbol{x}}_{t+1}, P_{t+1})$$

where $\boldsymbol{y}_{[0:t]} := (\boldsymbol{y}_0, ..., \boldsymbol{y}_t)$ *and* $\boldsymbol{a}_{[0:t]} := (\boldsymbol{a}_0, ..., \boldsymbol{a}_t)$.

i.e.

$$\mathbb{E}[x_{t+1} | F_{t+1}] = \bar{x}_{t+1}$$

AS GIVEN IN (MEASURE-1) ABOVE

PROOF: ASSUMING $[x_t \mid y_{[0:t]}, a_{[0:t-1]}] \sim N(\bar{x}_t, \bar{P}_t)$

$\therefore$

$$[x_{t+1} \mid y_{[0:t]}, a_{[0:t]}] = [A x_t + B a_t + \varepsilon_t \mid y_{[0:t]}, a_{[0:t]}]$$

$$\sim N(\underbrace{A\bar{x}_t + B a_t}_{\bar{x}_{t+1|t}}, \underbrace{A P_t A^T + \Sigma^{\varepsilon}}_{P_{t+1|t}})$$

LETS INCLUDE y_{t+1},

$$y_{t+1} = C x_{t+1} + \delta_{t+1}$$ ALSO GAUSSIAN.

$\therefore$

$$\mathbb{E}[y_{t+1} \mid y_{[0:t]}, a_{[0:t]}] = C \bar{x}_{t+1|t}$$

$$\mathbb{V}[y_{t+1} \mid y_{[0:t]}, a_{[0,t]}] = C P_{t+1|t} C^T + \Sigma^{\delta}$$

$$\mathbb{C}ov[x_{t+1}, y_{t+1} \mid y_{[0:t]}, a_{[0:t]}] = P_{t+1|t} C^T$$

$\therefore$

$$[x_{t+1}, y_{t+1} \mid y_{[0:t]}, a_{[0:t]}] \sim N\left((\bar{x}_{t+1|t}, C\bar{x}_{t+1|t}), \begin{bmatrix} P_{t+1|t} & P_{t+1|t}C^T \\ CP_{t+1|t} & CP_{t+1|t}C^T + \Sigma^\delta \end{bmatrix} \right)$$

THERE FORE [PROP 2.5 ii)]

$$[x_{t+1} \mid \underbrace{y_{[0:t+1]}}_{\text{NOW CONDITION ON } y_{t+1}}, a_{[0:t]}] \sim N\Big(\big(\bar{x}_{t+1|t} + P_{t+1|t}C^T[CP_{t+1|t}C^T + \Sigma^\delta]^{-1}(y_{t+1} - C\bar{x}_{t+1|t}\big),$$

$$P_{t+1|t} - P_{t+1|t}C^T[CP_{t+1|t}C^T + \Sigma^\delta]^{-1}CP_{t+1|t} \Big)$$

$$= N(\bar{x}_t, P_t) \quad \text{AS REQUIRED}$$

$\square$.