

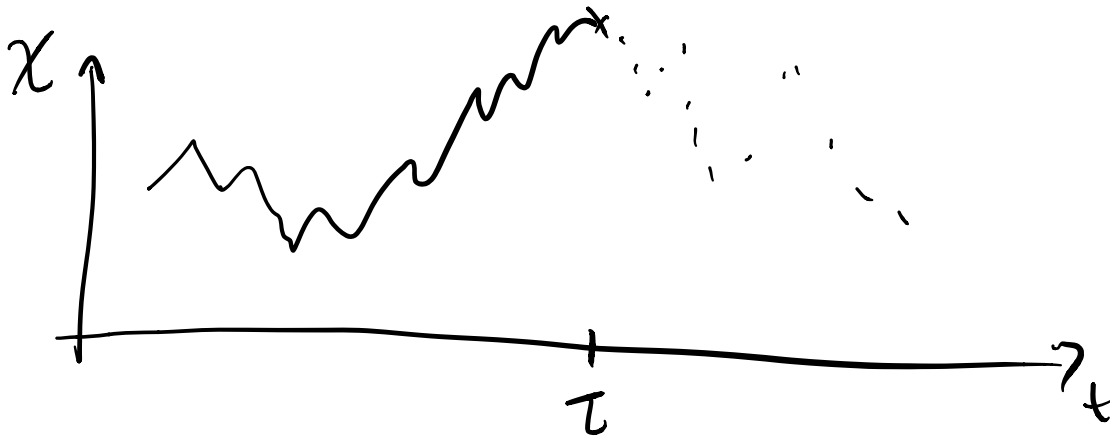
6. OPTIMAL STOPPING

DEFINITION:

- An Optimal Stopping Problem is an Markov Decision Process where there are two actions: $a = 0$ meaning to stop, and $a = 1$ meaning to continue. Here there are two types of costs

$$c(x, a) = \begin{cases} \kappa(x), & \text{for } a = 0 \quad (\text{the stopping cost}) \\ c(x), & \text{for } a = 1 \quad (\text{the continuation cost}), \end{cases}$$

This defines a *stopping problem*.



OBJECTIVE:

$$C(x) = \min_{\tau} \mathbb{E} \left[\sum_{t=0}^{\tau-1} c(x_t) + K(x_{\tau}) \right]$$

BELLMAN EQUATION:

Assuming that time is finite, the Bellman equation is

$$C_s(x) = \min \left\{ k(x), c(x) + \mathbb{E}_x[C_{s-1}(\hat{X})] \right\}$$

for $s \in \mathbb{N}$ and $C_0(x) = k(x)$.

OSLA & CLOSED SETS:

Def 44 (OLSA rule). In the one step lookahead (OLSA) rule we stop when ever $x \in S$ where

$$S = \{x : \underbrace{k(x)}_{\text{STOP NOW}} \leq c(x) + \underbrace{\mathbb{E}_x[k(\hat{X})]}_{\text{STOP NEXT}}\}.$$

We call S the stopping set. In words, you stop whenever it is better stop now rather than continue one step further and then stop.

Def 45 (Closed Stopping Set). We say the set $S \subset \mathcal{X}$ is closed, if once inside that said you cannot leave, i.e.

$$P_{xy} = 0, \quad \forall x \in S, y \notin S.$$

Prop 46. If, for the finite time stopping problem, the set S given by the one step lookahead rule is closed then the one step lookahead rule is an optimal policy.

[i.e. ONE STEP LEFT]

↓

PROOF: BY INDUCTION, IF $s=1$ THEN OSLA IS OPTIMAL ✓

SUPPOSE OSLA IS OPTIMAL FOR s , [WE SHOW ITS TRUE FOR $s+1$].

IF $x \notin S$, CLEARLY IS OPTIMAL TO CONTINUE [AS CONTINUING 1 MORE STEP & STOPPING IS BETTER!]

IF $x \in S$, THEN

$$L_{s+1}(x) = \min \{ k(x), c(x) + \mathbb{E}_x [C_s(\hat{x})] \}$$

$\hat{x} \in S$ SINCE S IS CLOSED

& SINCE OSLA IS OPTIMAL AT s STEPS

& $\hat{x} \in S$ OPTIMAL TO STOP

$$\therefore C_s(\hat{x}) = k(\hat{x})$$

$$= \min \{ k(x), c(x) + \mathbb{E}_x [k(\hat{x})] \} = k(x) \quad \therefore \text{OPTIMAL TO STOP IF } x \in S$$

SINCE $x \in S$

□.

OPTIMAL STOPPING IN INFINITE TIME:

Prop 47. If the following two conditions hold

- $K = \max_x k(x) < \infty, \min_x k(x) \geq 0$
- $C = \min_x c(x) > 0$
- S IS CLOSED.

then the One-Step-Lookahead-Rule is optimal.

PROOF: NOTICE IF WE CAN SHOW $L_s(x) \rightarrow L_\infty(x)$,
AS $L_s(x) = K(x) \forall s \forall x \in S$, THEN $L_\infty(x) = K(x) \forall x \in S$
 \therefore IS OPTIMAL TO STOP FOR $x \in S$.

τ^* OPTIMAL STOPPING TIME.

$$\underline{L_\infty(x)} \leq \underline{L_s(x)} \leq \underline{L_\infty(x) + K \mathbb{P}(\tau^* \geq s)}$$

STOP AT ANY
TIME

MUST STOP
BEFORE s

FOLLOW OPTIMAL POLICY
FOR INFINITE TIME THEN STOP AT s .

NOW LETS BOUND $P_x(\tau^* > s)$:

$$P_x(\tau^* \geq s) \leq \frac{1}{sC} \mathbb{E}_x \left[\underbrace{\left\{ \sum_{t=0}^{\tau^*-1} c(x_t) + K(x_{\tau^*}) \right\}}_{\text{THIS IS BIGGER THAN } sC} \mathbb{I}[\tau^* \geq s] \right]$$

$$\leq \frac{1}{sC} L_\infty(x) \leq \frac{K(x)}{sC} \leq \frac{K}{sC}$$

$\therefore P_x(\tau^* \geq s) \xrightarrow{s \rightarrow \infty} 0 \quad \therefore L_s(x) \rightarrow L_\infty(x) \quad \therefore \text{OSLA IS OPTIMAL} \quad \square.$

STOPPING A RANDOM WALK:

SUPPOSE X_t IS A RANDOM WALK ON $\{0, 1, 2, \dots, N\}$

i.e.

$$X_{t+1} = X_t + 1$$

WITH PROBABILITY $\frac{1}{2}$

$$X_{t+1} = X_t - 1$$

WITH PROBABILITY $\frac{1}{2}$

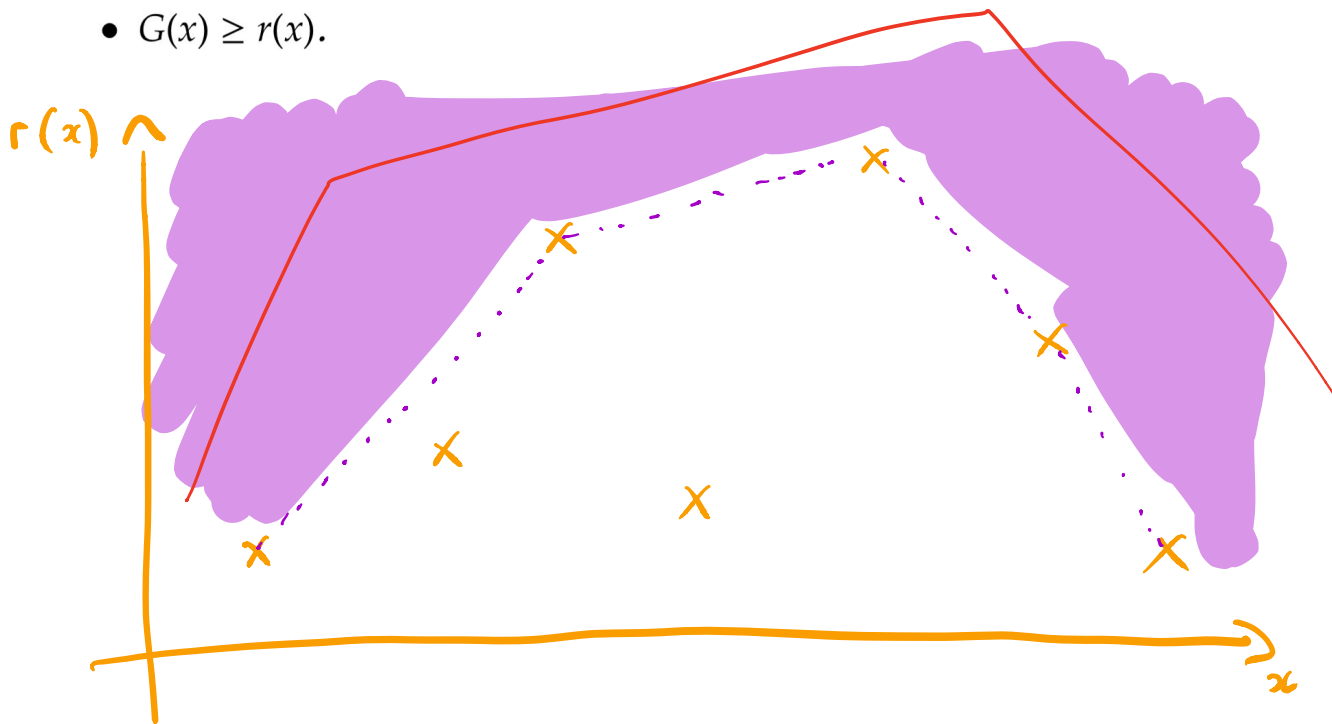
& PROCESS STOPS IF $X_t = 0$ OR $X_t = N$

YOU CAN DECIDE TO STOP. IF $X_t = x$
THEN REWARD IS $r(x) \geq 0$

STOPPING A RANDOM WALK:

Def 48 (Concave Majorant). For a function $r : \{0, \dots, N\} \rightarrow \mathbb{R}_+$ a concave majorant is a function G such that

- $G(x) \geq \frac{1}{2}G(x-1) + \frac{1}{2}G(x+1)$
- $G(x) \geq r(x)$.



Prop 49 (Stopping a Random Walk). Let X_t be a symmetric random walk on $\{0, \dots, N\}$ where the process is automatically stopped at 0 and N . For each $x \in \{0, \dots, N\}$, there is a positive reward of $r(x)$ for stopping. We are asked to maximize

$$\mathbb{E}[r(X_T)]$$

where T is our chosen stopping time. The optimal value function $V(x)$ is the minimal concave majorant, and that it is optimal to stop whenever $V(x) = r(x)$.

PROOF:

BELLMAN EQUATION IS

$$V(x) = \max \left\{ r(x), \frac{1}{2} V(x-1) + \frac{1}{2} V(x+1) \right\}$$

$\therefore V$ IS A CONCAVE MAJORANT.

PROOF (CONT): SUPPOSE $G(x)$ IS A CONCAVE MAJORANT.

LET $V_s(x)$ - BE s -STEP OPTIMAL SOLUTION

$$V_0(x) = r(x) \quad \therefore V_0(x) \leq G(x)$$

SUPPOSE [BY INDUCTION] $V_s(x) \leq G(x)$

THEN

$$V_{s+1}(x) = \max \left\{ r(x), V_s\left(\frac{x-1}{2}\right) + V_s\left(\frac{x+1}{2}\right) \right\}$$

$$\leq \max \left\{ r(x), G\left(\frac{x-1}{2}\right) + G\left(\frac{x+1}{2}\right) \right\}$$

$$\leq G(x) \quad \therefore V_s(x) \leq G(x) \quad \forall s$$

AS VALUE ITERATION CONVERGES $V(x) = \lim_{s \rightarrow \infty} V_s(x) \leq G(x)$

\therefore IS MINIMAL CONCAVE MAJORANT.

□.

