11. MERTON PORTFOLIO OPTIMIZATION

SINGLE ASSET CASE:

STOCK OBEYS: $dS_t = S_t \{ \sigma dB_t + \mu dt \}$, n_t = # OF STOCKS

BANK OBEYS: $dm_t = m_t \, r \, dt$ m_t = MONEY IN BANK

WEALTH OBEYS:

$$dW_t = \underbrace{r \underbrace{m_t}_{(W_t - n_t S_t)} dt}_{\text{MONEY IN BANK}} + \underbrace{n_t \, dS_t}_{\text{MONEY IN STOCKS}} - \underbrace{c_t \, dt}_{\text{CONSUMPTION.}}$$

THE MERTON PROBLEM:

$$V(w_0) = \max_{(n_t, c_t)_{t\geq 0} \in \mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right].$$

CONCAVE INCREASING
DIFFERENTIABLE FN.

THE BELLMAN EQUATION:

Prop 74. *The HJB equation for the Merton Problem can be written as*

$$0 = \max_c \{u(c) - c\partial_w V\} + \max_\theta \left\{\theta(\mu - r)\partial_w V + \frac{1}{2}\sigma^2\theta^2\partial_{ww}V\right\} - \rho V + rw\partial_w V$$

Here the optimal θ *and* c *are given by*

$$\theta^* = -\frac{\partial_w V}{\partial_{ww}V}\sigma^{-2}(\mu - r), \qquad c^* = (u')^{-1}(\partial_w V)$$

PROOF:

$$dW_t = r(W_t - n_t S_t)\,dt + \underbrace{n_t\, dS_t}_{n_t S_t \mu dt + n_t S_t \sigma dB_t} - c_t\, dt$$

$$= (rW_t + (\mu - r)\theta_t - c_t)dt + \theta_t \sigma dB_t$$

$$\left[0 = \max_a \left\{ \text{"INSTANTANEOUS COST"} - \rho V + \text{"DRIFT FROM ITÔ'S FORMULA"} \right\} \right]$$

PROOF:

$$dV(W_t) = \partial_w V(W_t)\,dW_t + \frac{1}{2}\partial_{ww}V(W_t)\,d[W_t]$$

$$= \partial_w V(W_t)\left[r(W_t - n_t S_t)dt + n_t\,dS_t - c_t\,dt\right] + \frac{\theta^2\sigma^2}{2}\partial_{ww}V(W_t)\,dt$$

$\therefore$ HJB IS

$$0 = \max_{\theta, c}\left\{u(c) - \rho V + \left(rw + \theta(\mu - r) - c\right)\partial_w V + \frac{1}{2}\sigma^2\theta^2\partial_{ww}V\right\}$$

$$= \underbrace{\max_{c}\left\{u(c) - c\,\partial_w V\right\}}_{\Rightarrow\ u'(c) = \partial_w V} + \underbrace{\max_{\theta}\left\{\theta(\mu - r) + \frac{1}{2}\sigma^2\theta^2\partial_{ww}V\right\}}_{\Rightarrow -(\mu - r)\partial_w V = \sigma^2\theta\,\partial_{ww}V} - \rho V + rw\,\partial_w V$$

$\square$

[GIVES THE CONDITION ON θ, c ABOVE]

PROP IS GOOD BUT WE NEED TO FIND $\partial_w V$, $\partial_{ww} V$.

MERTON FOR CRRA UTILITY

FOR $R>0$

$$u(c) = \frac{c^{1-R}}{1-R}$$

$$\left(= \log c \quad \text{IF } R=1 \right)$$

$\leftarrow$ CONSTANT RELATIVE RISK AVERSION (CRRA)

[MEANS SAME ATTITUDE TO RISK IN A BET IF YOU MULTIPLY THE STAKES]

SO WE SOLVE FOR

$$V(w_0) = \max_{(n_t, c_t)_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt\right].$$

Prop 75. *For a CRRA utility it holds that:*
a) The Value function takes the form

$$V(w) = \gamma \frac{w^{1-R}}{1-R}$$

SO WE CAN DIFFERENTIATE $V(\cdot)$

for some position constant $\gamma > 0$.
b) The HJB equation is optimized by

& GET θ^* & c^*

$$\theta^* = \frac{w}{R}\sigma^{-2}(\mu - r),$$

$$c^* = \gamma^{-\frac{1}{R}} w \quad \text{and} \quad \sup_c \{u(c) - c\partial_w V\} = \frac{R}{1-R}\gamma^{1-\frac{1}{R}} w^{1-R}.$$

c) The HJB equation is satisfied by parameters

$$\gamma^* = R^{-1}\left\{\rho + (R-1)\left(r + \frac{1}{2}\frac{\kappa^2}{R}\right)\right\}$$

where

$$\kappa = \sigma^{-1}(\mu - r).$$

PROOF: a)

$$V(\lambda w) = \max_{(n_t, c_t)_{t \geq 0} \in \mathcal{P}(\lambda w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt\right]$$

$$= \max_{(n_t, c_t)_{t \geq 0} \in \mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{(\lambda c_t)^{1-R}}{1-R} dt\right] = \lambda^{1-R} V(w).$$

NOW LET $\lambda = \frac{1}{w}$ & $\gamma = (1-R)V(1)$ GIVES $V(w) = \gamma \frac{w^{1-R}}{1-R}$

b) $\partial_w V = \gamma w^{-R}$, $\partial_{ww} V = -R\gamma w^{-R-1}$

$$\theta^* = -\frac{\partial_w V}{\partial_{ww} V} \sigma^{-2}(\mu - r) = \frac{w}{R} \frac{(\mu - r)}{\sigma^2}$$

[FROM PREVIOUS PROP]

ALSO

$$\underset{\substack{\| \\ c^{-R}}}{u'(c)} = \partial_w V = \gamma w^{-R} \Rightarrow c = \gamma^{-\frac{1}{R}} w$$

PROOF (CONT)

$$\sup_c \{u(c) - c\partial_w V\} = \frac{c^{*1-R}}{1-R} - c^* \partial_w V(c^*)$$
$$= \frac{(\gamma^{-\frac{1}{R}} w)^{1-R}}{1-R} - (\gamma^{-\frac{1}{R}} w)\gamma w^{-R}$$
$$= \frac{R}{1-R}\gamma^{1-\frac{1}{R}} w^{1-R},$$

A BORING CALCULATION!

SUBSTITUTE INTO HJB EQN FOR γ:

$$0 = \frac{R}{1-R}\gamma^{1-\frac{1}{R}} w^{1-R} - \frac{1}{2}\sigma^{-2}(\mu - r)^2 \frac{(\partial_w V)^2}{\partial_{ww} V} - \rho V + rw\partial_w V$$
$$= \frac{R}{1-R}\gamma^{1-\frac{1}{R}} w^{1-R} - \frac{1}{2}\sigma^{2}(\mu - r)^2 \frac{(\gamma w^{-r})^2}{(-R\gamma w^{-R-1})} - \rho\gamma \frac{w^{1-R}}{1-R}$$
$$= \gamma w^{1-R}\left[\frac{R}{1-R}\gamma^{\frac{1}{R}} + \frac{1}{2}\sigma^2 \frac{(\mu - r)^2}{R} - \frac{\rho}{1-R} + r\right].$$

A QUICK SUMMARY SO FAR:

To summarize: we notice we have shown that the parameters

$$\theta^* = \frac{w}{R}\sigma^{-2}(\mu - r), \quad c^* = \gamma^{-\frac{1}{R}} w, \tag{2.7a}$$

$$\gamma^* = R^{-1}\left\{\rho + (R-1)\left(r + \frac{1}{2}\frac{\kappa^2}{R}\right)\right\}, \quad \kappa = \sigma^{-1}(\mu - r). \tag{2.7b}$$

&

$$W_t = W_0 \exp\left\{R^{-1}\kappa B_t + (r + \frac{1}{R^2}\kappa^2(2R_1) - \gamma)t\right\}$$

&

Thrm 97. *The parameters in* (2.7), *above, are optimal for the Merton problem.*

[SEE NOTES FOR A PROOF]

MULTIPLE ASSETS:

[SAME AS BEFORE JUST A BIT MORE MESSY].

IF

$$dS_t^i = S_t^i \left\{ \sum_{j=1}^{N} \sigma^{ij} dB_t^j + \mu^i dt \right\}, \quad i = 1, \ldots, d$$

FOR

$$dW_t = r\left(W_t - \boldsymbol{n}_t \cdot \boldsymbol{S}_t\right) dt + \boldsymbol{n}_t \cdot d\boldsymbol{S}_t - c_t dt$$

THEN HJB

$$0 = \max_{\theta, c} \left\{ \underbrace{u(t,c)}_{\text{CRRA AGAIN.}} + \partial_t V + \left(rw + \theta \cdot (\mu - r) - c\right) \partial_w V + \frac{1}{2} |\sigma^\top \theta|^2 \partial_{ww} V \right\}.$$

& OPTIMAL PARAMETERS ARE

$$\boldsymbol{\theta}^* = \frac{w}{R} (\sigma\sigma^\top)^{-1} (\boldsymbol{\mu} - \boldsymbol{r}), \quad c^* = \gamma^{-\frac{1}{R}} w$$

$$\gamma^{-\frac{1}{R}} = R^{-1} \left\{ \rho + (R-1)(r + \frac{1}{2} \frac{|\boldsymbol{\kappa}|^2}{R}) \right\} \qquad \boldsymbol{\kappa} = \sigma^{-1} (\boldsymbol{\mu} - \boldsymbol{r}).$$

GENERAL UTILITIES:

$$0 = \max_{\theta,c}\left\{\underbrace{u(t,c)}_{\substack{\text{INCREASING}\\ \text{CONCAVE}\\ \lim_{c\to\infty} u'(t,c)=0}} + \partial_t V + (rw + \theta\cdot(\mu - r) - c)\,\partial_w V + \frac{1}{2}|\sigma^\intercal\theta|^2\partial_{ww}V\right\}.$$

LET $u^*(t,z) = \max_c\{u(t,c) - zc\}$ & $J(t,z) = V(t,w) - wz$ FOR $z = \partial_w V(t,w)$.

Thrm 99. *The HJB equation can be written as*

$$0 = u^*(t,c) + \partial_t J - rz\partial_z J + \frac{1}{2}|\kappa|^2 z^2\partial_{zz}J$$

Moreover if we suppose that $u(t,x) = e^{-\rho t}u(x)$, for $u(x)$ concave and increasing, the HJB equation becomes

$$0 = u^*(y) - \rho j(y) + (\rho - r)yj'(y) + \frac{1}{2}|\kappa|^2y^2j''(y)$$

PROOF: FIRST WE SHOW $\partial_z J = -w$ ✓, $\partial_{zz} J = -\dfrac{1}{\partial_{ww} V}$ ✓ $\quad \partial_t J = \partial_t V$ ✓

$$J(z) = V(\underbrace{V'^{-1}(z)}_{=w}) - z\underbrace{V'^{-1}(z)}_{=w}$$

$$\therefore J'(z) = \cancel{\frac{d}{dz}(V'^{-1}(z)) \cdot} \cancel{\underbrace{V'(V'^{-1}(z))}_{z}} - \cancel{z\frac{d}{dz}V'^{-1}(z)} - \underbrace{V'^{-1}(z)}_{=w} = -w \quad ✓$$

&

$$\therefore J''(z) = -\frac{d}{dz}(V')^{-1}(z) = -\frac{1}{V''(V'^{-1}(z))} = -\frac{1}{V''(w)} \quad ✓$$

&

$$\frac{\partial J}{\partial t} = \partial_t V(t,w) + \cancel{\frac{dw}{dt}\partial_w V} - \cancel{z\frac{dw}{dt}} = \partial_t V \quad ✓$$

HJB IS

$$0 = \max_c \left\{ u(t,z) - c\,\partial_w V \right\} + \max_\theta \left\{ \theta\cdot(\mu - r)\partial_w V + \frac{1}{2}|\sigma\theta|^2 \partial_{ww} V \right\} + \partial_t V + rw\,\partial_w V$$

$$= u^*(t, \underbrace{\partial_w V}_{z}) - \frac{1}{2}|K|^2 \underbrace{(\partial_w V)^2}_{z^2} \frac{1}{\underbrace{\partial_{ww} V}_{\partial_{zz} J}} + \underbrace{\partial_t V}_{\partial_t J} + r\underbrace{w}_{-\partial_z J}\underbrace{\partial_w V}_{z}$$

$$= u^*(t,z) + \frac{1}{2}|K|^2 z^2 \partial_{zz} J + \partial_t J - rz\,\partial_z J.$$

PROOF (CONT)

[SIMILAR TO CRRA ARGUMENT]

CAN CHECK IF $u(t,x) = e^{-\rho t}u(x)$ THEN $V(t,w) = e^{-\rho t}V(w)$

LETTING $j(z) = V(w) - wz$, $z = \partial_w V(t,w)$

THEN, DIFFERENTIATING GIVES

$$J(t,z) = e^{-\rho t} j(y), \qquad \partial_t J = -\rho e^{-\rho t} j(y) + \rho e^{-\rho t} y j'(y)$$
$$\partial_z J = j'(y), \qquad \partial_{zz} J = e^{\rho t} j''(y)$$

SUBSTITUTING INTO HJB EQUATION ABOVE GIVES:

$$0 = \underbrace{u^*(t,z)}_{=e^{-\rho t}u^*(y)} + \underbrace{\partial_t J}_{=-\rho e^{-\rho t} j(y) + \rho e^{-\rho t} y j'(y)} - \underbrace{rz\partial_z J}_{re^{-\rho t} y j'(y)} + \underbrace{\frac{1}{2}|\kappa|^2 z^2 \partial_{zz} J}_{\frac{1}{2}|\kappa|^2 y^2 e^{-\rho t} j''(y)}$$

$\square$

FURTHER

EXAMPLES

EXAMPLE 1 : CARA UTILITY

A UTILITY FUNCTION HAS CONSTANT ABSOLUTE RISK AVERSION IF

$$u(c) = -\frac{1}{R} e^{-Rc}$$

YOUR ATTITUDE TO RISK DOESN'T CHANGE WITH THE SIZE OF YOUR WEALTH.

MERTON PORTFOLIO FOR CARA UTILITY

CLAIM: IF YOU HAVE A CARA UTILITY $u(c) = -\frac{1}{R} e^{-Rc}$

THEN $V(w) = -A e^{-rRw}$ FOR SOME A

& $\theta = \frac{1}{Rr} \left(\frac{\mu - r}{\sigma^2} \right)$, $c = rAw - \log(rRA)$

KEEP A CONSTANT AMOUNT IN RISKY ASSETS.

SOLUTION: WE TAKE $V(w) = -A e^{-rRw}$

[ITS A GUESS BUT WITH GOOD REASON]

[i.e. MORE MONEY SHOULD GET INVESTED IN RISK FREE WITH INTEREST r.]

WE KNOW THAT

$$\theta^* = -\frac{\partial_w V}{\partial_{ww} V}\left(\frac{\mu - r}{\sigma^2}\right) \quad \& \quad u'(c^*) = \partial_w V$$

$$\partial_w V = ArR\, e^{-rRw}, \quad \partial_{ww} V = -Ar^2R^2 e^{-Rw}, \quad u'(c) = e^{-Rc}$$

$\therefore$

$$\theta^* = -\frac{1}{rR}\left(\frac{\mu - r}{\sigma^2}\right)$$

$$u'(c^*) = e^{-Rc^*} = ArR\, e^{-rRw} \Rightarrow c^* = rw - \log(rAR)$$

SUBSTITUTING IN TO HJB GIVES A. $\square$.

Example 2: Interest Rate Risk

Ex 78. *We consider the Merton investment problem but now the interest rate can vary. Wealth* $(W_t : t \geq 0)$ *satisfies* $W_0 = w$ *and obeys the stochastic differential equation*

$$dW_t = \{r_t W_t + (\mu - r_t)\theta_t - c_t\}dt + \theta_t \sigma dB_t.$$

and the interest rate obeys the stochastic differential equation

$$dr_t = \beta(\bar{r} - r_t)dt + \sigma_r dB_t^r$$

where β, $\bar{r}$ *and* σ_r *are fixed parameters and* B_t^r *is a standard Brownian motion with covariation* $[B_t^r, B_t] = \eta t$.

We maximize the utility of a CRRA utility function:

$$V(w,r) = \max_{(\theta_s, c_s)_{s\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-\rho s} u(c_s) ds \,\middle|\, W_0 = w, r_0 = r\right], \quad \text{with} \quad u(c_s) = \frac{c_s^{1-R}}{1-R}.$$

Show that,

$$V(w,r) = \gamma(r)\frac{w^{1-R}}{1-R}$$

for some function $\gamma(r)$, *and analyse the HJB equation to find the differential equation that the function* $\gamma(r)$ *must satisfy.*

SOLUTION

Ans 117. *First note that*

$$V(\lambda w, r) = \max_{(n_t, c_t)_{t\geq 0} \in \mathcal{P}(\lambda w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt\right]$$
$$= \max_{(n_t, c_t)_{t\geq 0} \in \mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{(\lambda c_t)^{1-R}}{1-R} dt\right] = \lambda^{1-R} V(w, r).$$

Setting $\lambda = w^{-1}$ *and* $\gamma(r) = (1-R)V(1, r)$ *gives*

$$V(w, r) = \gamma(r) \frac{w^{1-R}}{1-R}.$$

Ito's formula gives

$$dV(W_t, r_t) = \partial_w V dW_t + \partial_r V dr_t + \frac{1}{2}\partial_{w,w} V d[W]_t + \partial_{w,r} V d[W, r]_t + \frac{1}{2}\partial_{r,r} V d[r]_t$$
$$= \partial_w V \cdot [\{r_t W_t + (\mu - r_t)\theta_t - c_t\}dt + \theta_t \sigma dB_t]$$
$$+ \partial_r V \cdot \left[\beta(\bar{r} - r_t)dt + \sigma_r dB_t^r\right]$$
$$+ \frac{1}{2}\partial_{w,w} V \cdot \theta_t^2 \sigma^2 dt + \partial_{w,r} \eta dt + \frac{1}{2}\partial_{r,r} \sigma_r^2 dt$$

The HJB equation is given by

$$
\begin{aligned}
0 &= \max_{c,\theta}\left\{u(c) - \rho V + \frac{\mathbb{E}[dV(W_t, r_t)]}{dt}\right\} \\
&= \max_{c,\theta}\left\{u(c) - \rho V + \frac{1}{2}\sigma^2\theta^2\partial_{ww}V + \sigma\sigma_r\eta\theta\partial_{wr}V + \frac{1}{2}\sigma_r^2\partial_{rr}V\right. \\
&\qquad\qquad \left. + (rw + \theta(\mu - r) - c)\partial_w V + \beta(\bar{r} - r)\partial_r V\right\}.
\end{aligned}
$$

The HJB equation gives $V(w,r) = \gamma(r)u(w)$. Optimizing the HJB equation for c gives

$$
\max_c\left\{u(c) - c\partial_w V\right\} \quad \Longrightarrow \quad u'(c) = \partial_w V \quad \Longrightarrow \quad c = w\gamma(r)^{-1/R}.
$$

Optimizing the HJB equation for θ gives,

$$
\begin{aligned}
&\sigma^2\theta\partial_{ww}V + \sigma\sigma_r\eta\partial_{wr}V + (\mu - r)\partial_w V = 0 \\
\Longrightarrow\ & -\sigma^2\theta R w^{-R-1}\gamma(r) + \sigma\sigma_r\eta w^{-R}\gamma'(r) + (\mu - r)\gamma(r)w^{-R} = 0 \\
\Longrightarrow\ & \theta = w \cdot \frac{\sigma\sigma_r\eta\gamma'(r) + (\mu - r)\gamma(r)}{\sigma^2 R\gamma(r)}
\end{aligned}
$$

Substituting these answers into the HJB equation gives

Substituting these answers into the HJB equation gives

$$
\begin{aligned}
0 =& \max_c \left\{ u(c) - c\partial_w V \right\} + \max_\theta \left\{ \frac{1}{2}\sigma^2\theta^2\partial_{ww}V + \sigma\sigma_r\eta\theta\partial_{wr}V + \theta(\mu - r)\partial_w V \right\} \\
&- \rho V + \frac{1}{2}\sigma_r^2\partial_{rr}V + (rw + \beta(\bar{r} - r))\partial_r V \\
=& u(w)R\gamma^{1-1/R} + u(w)\frac{(1-R)}{R}\frac{(\sigma\sigma_r\eta\gamma' + (\mu - r))^2}{2\sigma^2} \\
&- \rho u(w)\gamma + \frac{1}{2}\sigma_r^2 u(w)\gamma''(r) + (rw - \beta(\bar{r} - r))u(w)\gamma(r)
\end{aligned}
$$

which gives the required ODE for $\gamma = \gamma(r)$:

$$
0 = R\gamma^{1-1/R} + \frac{(1-R)}{R}\frac{(\sigma\sigma_r\eta\gamma' + (\mu - r))^2}{2\sigma^2} - \rho\gamma + \frac{1}{2}\sigma_r^2\gamma'' + (rw - \beta(\bar{r} - r))\gamma
$$