10. DIFFUSION CONTROL.

DIFFUSION CONTROL PROBLEM

FOR

$$dX_t = \mu_t(X_t, a_t)dt + \sigma_t(X_t, a_t) \cdot dB_t$$

MINIMIZE

$$L(x_0) := \underset{\underline{\Pi \in \mathcal{P}}}{minimize}\ C(x_0, \Pi) := \mathbb{E}_{x_0}\left[\int_0^T e^{-\alpha t} c_t(X_t, \pi_t)dt + e^{-\alpha T} c_T(X_T)\right] \text{ (DCP)}$$

POLICIES

THE HJB EQUATION.

Def 67 (Hamilton-Jacobi-Bellman Equation). *For a Diffusion Control Problem* (DCP), *the equation*

$$0 = \min_{a \in \mathcal{A}} \left\{ c_t(x,a) + \partial_t L_t(x) + \mu_t(x,a) \cdot \partial_x L_t(x) + \frac{1}{2}[\sigma^T \sigma] \cdot \partial_{xx} L_t(x) - \alpha L_t(x). \right\} \tag{HJB}$$

INTUITIVE DEFINITION:

$$0 = \min_{a} \left\{ \text{"INSTANTANEOUS COST"} + \text{"DRIFT FOR ITO'S FORMULA"} \right\}$$

DERIVING HJB EQUATION

TAYLOR:

$$X_{t+\delta} - X_t = \mu_t(X_t, \pi_t)\delta + \sigma_t(X_t, \pi_t)(B_{T+\delta} - B_t)$$

COSTS:

$$C_t(x, \Pi) \approx \mathbb{E}\left[\sum_{t \in \{0, \delta, \ldots, T-\delta\}} (1 - \alpha\delta)^{\frac{t}{\delta}} c_t(X_t, \pi_t)\delta + (1 - \alpha\delta)^{\frac{T}{\delta}} c_T(X_T) \right].$$

NOW IS AN M.D.P.

BELLMAN EQN IS

$$L_t(x) = \min_{a\in\mathcal{A}} \left\{ c_t(x,a)\delta + (1-\alpha\delta)\mathbb{E}_{x,a}\left[L_{t+\delta}(X_{t+\delta})\right]\right\}.$$

THUS

$$0 = \min_{a\in\mathcal{A}} \left\{ c_t(x,a) + \frac{1}{\delta}\mathbb{E}_{x,a}\left[\underline{L_{t+\delta}(X_{t+\delta}) - L_t(x)}\right] - \alpha\mathbb{E}_{x,a}\left[L_{t+\delta}(X_{t+\delta})\right]\right\}.$$

EXPAND.

ITÔ'S FORMULA

$$\begin{aligned} &L_{t+\delta}(X_{t+\delta}) - L_t(X_t) \\ &\approx \left[\partial_t L + \mu_t(X_t,\pi_t)\cdot\partial_x L + \frac{\sigma_t(X_t,\pi_t)^2}{2}\partial_{xx}L\right]\delta + \partial_x L\cdot\sigma_t(X_t,\pi_t)\cdot(B_{t+\delta}-B_t) \end{aligned}$$

THUS

$$\frac{1}{\delta}\mathbb{E}_{x,a}\left[L_{t+\delta}(X_{t+\delta}) - L_t(x)\right] = \partial_t L + \mu_t(X_t,\pi_t)\cdot\partial_x L + \frac{\sigma_t(X_t,\pi_t)^2}{2}\partial_{xx}L$$

SUBSTITUTING BACK GIVES:

$$0 = \min_{a\in\mathcal{A}}\left\{c_t(x,a) + \partial_t L + \mu_t(x,a)\cdot\partial_x L + \frac{\sigma_t(x,a)^2}{2}\partial_{xx}L - \alpha L_t(x)\right\}.$$

AKA. THE HJB EQN.

A MARTINGALE PRINCIPLE:

Thrm 68 (Davis-Varaiya Martingale Prinicple of Optimality). *Suppose that there exists a function $L_t(x)$ with $L_T(x) = e^{-\alpha T} c_T(x)$ and such that for any policy Π with states X_t*

$$M_t = L_t(X_t) + \int_0^t e^{-\alpha\tau} c_\tau(X_\tau, \Pi) d\tau$$

is a sub-martingale and, moreover that for some policy Π^*, M_t is a martingale then Π^* is optimal and

$$L_0(X_0) = \min_{\Pi\in\mathcal{P}} \mathbb{E}\left[\int_0^T e^{-\alpha\tau} c_\tau(X_\tau, \pi_\tau) d\tau + c_T(X_T)\right].$$

PROOF:

$$L_0(x) = M_0 \leq \mathbb{E}_x[M_T] = \mathbb{E}_x\left[\int_0^T e^{-\alpha s} c_s(X_s, \pi_s) ds + \underbrace{L_T(X_T)}_{e^{-\alpha T} c_T(X_T)}\right]$$

$$\therefore \quad L_0(x_0) \leq C(x, \pi) \quad \forall \pi$$

& IF M_y FOR π^* THEN

$$L_0(x_0) = C(x, \pi^*) \quad \text{FOR } \pi^*$$

$\therefore \pi^*$ IS OPTIMAL $\square$.