9. STOCHASTIC INTEGRATION

(RECAP).

TO DERIVE THE HJB EQUATION WE NEEDED

- TAYLOR EXPANSIONS
- CHAIN RULE.

WE RECAP THE SAME FOR STOCHASTIC INTEGRAL

STOCHASTIC INTEGRAL FOR BROWNIAN MOTION

INDEPENDENT, GAUSSIAN INCREMENTS, LINE VARIATION.

SLLN.

$$\sum_{t\in\{0,\delta,..,T\}} (B_{t+\delta} - B_t) = B_T \quad \textbf{and} \quad \sum_{t\in\{0,\delta,..,T\}} (B_{t+\delta} - B_t)^2 \approx T$$

So

STOCHASTIC INTEGRAL

$$\sum_{t\in\{0,\delta,..,T\}} \sigma(X_t)\,(B_{t+\delta} - B_t) \approx \int_0^T \sigma(X_t)dB_t$$

$$\sum_{t\in\{0,\delta,..,T\}} \mu(X_t)\,(B_{t+\delta} - B_t)^2 \approx \int_0^T \mu(X_t)dt.$$

STOCHASTIC DIFFERENTIAL EQUATIONS

DEFINE

$$X_{t+\delta} - X_t = \sigma(X_t)(B_{t+\delta} - B_t) + \mu(X_t)\delta, \qquad t = 0, \delta, 2\delta, \ldots.$$

LET $\delta \to 0$

$$X_T = X_0 + \int_0^T \sigma(X_t)dB_t + \int_0^T \mu(X_t)dt.$$

THIS GIVES US "TAYLOR" APPROXIMATIONS FOR S.D.Es.

ITô's FORMULA

FOR OUR SDE FROM BEFORE:

$$
\begin{aligned}
& f(X_{t+\delta}) - f(X_t) \\
=& f(X_t + \sigma(X_t)(B_{t+\delta} - B_t) + \mu(X_t)\delta) - f(X_t) \\
=& f'(X_t)\left\{\mu\delta + \sigma\cdot(B_{t+\delta} - B_t)\right\} + \frac{f''(X_t)}{2}\left\{\mu\delta + \sigma\cdot(B_{t+\delta} - B_t)\right\}^2 + o(\delta) \\
=& f'(X_t)\left\{\mu\delta + \sigma\cdot(B_{t+\delta} - B_t)\right\} + \frac{f''(X_t)}{2}\sigma^2\cdot(B_{t+\delta} - B_t)^2 + o(\delta)
\end{aligned}
$$

So

$$f(X_{t+\delta}) - f(X_t) \approx \left[f'(X_t)\mu(X_t) + \frac{\sigma(X_t)^2}{2}f''(X_t)\right]\delta + f'(X_t)\sigma(X_t)\,(B_{t+\delta} - B_t)$$

So

$$f(X_T) - f(X_0) = \int_0^T \left[f'(X_t)\mu(X_t) + \frac{\sigma(X_t)^2}{2}f''(X_t)\right]dt + \int_0^T f'(X_t)\sigma(X_t)dB_t.$$

THIS GIVES US A "CHAIN RULE" FOR S.D.E.s