11. MERTON PORTFOLIO OPTIMIZATION

SINGLE ASSET CASE:

STOCK OBEYS: $dS_t = S_t \{ \sigma \, dB_t + \mu \, dt \}$, $\quad n_t = \#$ OF STOCKS

BANK OBEYS: $dm_t = m_t \, r \, dt$ $\quad m_t =$ MONEY IN BANK

WEALTH OBEYS:

$$dW_t = r \underbrace{m_t}_{\underbrace{(W_t - n_t S_t)}_{\text{MONEY IN BANK}}} dt + \underbrace{n_t \, dS_t}_{\text{MONEY IN STOCKS}} - \underbrace{c_t \, dt}_{\text{CONSUMPTION.}}$$

THE MERTON PROBLEM:

$$V(w_0) = \max_{(n_t, c_t)_{t \geq 0} \in \mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right].$$

CONCAVE INCREASING
DIFFERENTIABLE FN.

THE BELLMAN EQUATION:

Prop 74. *The HJB equation for the Merton Problem can be written as*

$$0 = \max_c \{u(c) - c\partial_w V\} + \max_\theta \left\{\theta(\mu - r)\partial_w V + \frac{1}{2}\sigma^2\theta^2\partial_{ww}V\right\} - \rho V + rw\partial_w V$$

Here the optimal θ *and* c *are given by*

$$\theta^* = -\frac{\partial_w V}{\partial_{ww}V}\sigma^{-2}(\mu - r), \qquad c^* = (u')^{-1}(\partial_w V)$$

PROOF:

$$dW_t = r(W_t - n_t S_t)\,dt + \underbrace{n_t\, dS_t}_{n_t S_t \mu\, dt + n_t S_t \sigma\, dB_t} - c_t\, dt$$

$$= (rW_t + (\mu - r)\theta_t - c_t)dt + \theta_t \sigma\, dB_t$$

$$\left[\, 0 = \max_a \{\text{"INSTANTANEOUS COST"} - \rho V + \text{"DRIFT FROM ITO'S FORMULA"}\} \,\right]$$

PROOF:

$$dV(W_t) = \partial_w V(W_t)\, dW_t + \frac{1}{2} \partial_{ww} V(W_t)\, d[W_t]$$

$$= \partial_w V(W_t)\left[r(W_t - n_t S_t)dt + n_t\, dS_t - c_t\, dt\right] + \frac{\theta^2 \sigma^2}{2} \partial_{ww} V(W_t)\, dt$$

∴ HJB IS

$$0 = \max_{\theta, c} \left\{ u(c) - \rho V + (rw + \theta(\mu - r) - c)\partial_w V + \frac{1}{2}\sigma^2 \theta^2 \partial_{ww} V \right\}$$

$$= \underbrace{\max_{c} \left\{ u(c) - c\,\partial_w V \right\}}_{\Rightarrow\ u'(c) = \partial_w V} + \underbrace{\max_{\theta} \left\{ \theta(\mu - r) + \frac{1}{2}\sigma^2 \theta^2 \partial_{ww} V \right\}}_{\Rightarrow\ -(\mu - r)\partial_w V = \sigma^2 \theta\, \partial_{ww} V} - \rho V + rw\, \partial_w V$$

GIVES THE CONDITION ON θ, c ABOVE

$\square$

PROP IS GOOD BUT WE NEED TO FIND $\partial_w V$, $\partial_{ww} V$.

MERTON FOR CRRA UTILITY

FOR $R>0$

$$u(c) = \frac{c^{1-R}}{1-R}$$

$$\left(= \log c \quad \text{IF } R=1 \right)$$

← CONSTANT RELATIVE RISK AVERSION (CRRA)

[MEANS SAME ATTITUDE TO RISK IN A BET IF YOU MULTIPLY THE STAKES]

SO WE SOLVE FOR

$$V(w_0) = \max_{(n_t, c_t)_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt\right].$$

Prop 75. *For a CRRA utility it holds that:*
a) The Value function takes the form

$$V(w) = \gamma \frac{w^{1-R}}{1-R}$$

SO WE CAN DIFFERENTIATE $V(\cdot)$

for some position constant $\gamma > 0$.
b) The HJB equation is optimized by

& GET θ^* & c^*

$$\theta^* = \frac{w}{R}\sigma^{-2}(\mu - r),$$

$$c^* = \gamma^{-\frac{1}{R}} w \quad \textit{and} \quad \sup_c \{u(c) - c\partial_w V\} = \frac{R}{1-R}\gamma^{1-\frac{1}{R}} w^{1-R}.$$

c) The HJB equation is satisfied by parameters

$$\gamma^* = R^{-1}\left\{\rho + (R-1)\left(r + \frac{1}{2}\frac{\kappa^2}{R}\right)\right\}$$

where

$$\kappa = \sigma^{-1}(\mu - r).$$

PROOF: a)

$$V(\lambda w) = \max_{(n_t,c_t)_{t\geq 0}\in\mathcal{P}(\lambda w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-R}}{1-R}dt\right]$$

$$= \max_{(n_t,c_t)_{t\geq 0}\in\mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{(\lambda c_t)^{1-R}}{1-R}dt\right] = \lambda^{1-R}V(w).$$

NOW LET $\lambda = \frac{1}{w}$ & $\gamma = (1-R)V(1)$ GIVES $V(w) = \gamma\frac{w^{1-R}}{1-R}$

b) $\partial_w V = \gamma w^{-R}$, $\partial_{ww} V = -R\gamma w^{-R-1}$

$$\theta^* = -\frac{\partial_w V}{\partial_{ww} V}\sigma^{-2}(\mu - r) = \frac{w}{R}\frac{(\mu-r)}{\sigma^2}$$

[FROM PREVIOUS PROP]

ALSO

$$\underset{\overset{\shortparallel}{c^{-R}}}{u'(c)} = \partial_w V = \gamma w^{-R} \Rightarrow c = \gamma^{-\frac{1}{R}} w$$

PROOF (CONT)

$$\left.\begin{aligned}\sup_c \{u(c) - c\partial_w V\} &= \frac{c^{*1-R}}{1-R} - c^*\partial_w V(c^*)\\ &= \frac{(\gamma^{-\frac{1}{R}}w)^{1-R}}{1-R} - (\gamma^{-\frac{1}{R}}w)\gamma w^{-R}\\ &= \frac{R}{1-R}\gamma^{1-\frac{1}{R}}w^{1-R},\end{aligned}\right\}$$

A BORING CALCULATION!

SUBSTITUTE INTO HJB EQN FOR γ:

$$\begin{aligned}0 &= \frac{R}{1-R}\gamma^{1-\frac{1}{R}}w^{1-R} - \frac{1}{2}\sigma^{-2}(\mu - r)^2\frac{(\partial_w V)^2}{\partial_{ww}V} - \rho V + rw\partial_w V\\ &= \frac{R}{1-R}\gamma^{1-\frac{1}{R}}w^{1-R} - \frac{1}{2}\sigma^2(\mu - r)^2\frac{(\gamma w^{-r})^2}{(-R\gamma w^{-R-1})} - \rho\gamma\frac{w^{1-R}}{1-R}\\ &= \gamma w^{1-R}\left[\frac{R}{1-R}\gamma^{\frac{1}{R}} + \frac{1}{2}\sigma^2\frac{(\mu - r)^2}{R} - \frac{\rho}{1-R} + r\right].\end{aligned}$$

A QUICK SUMMARY SO FAR:

To summarize: we notice we have shown that the parameters

$$\theta^* = \frac{w}{R}\sigma^{-2}(\mu - r), \quad c^* = \gamma^{-\frac{1}{R}} w, \tag{2.7a}$$

$$\gamma^* = R^{-1}\left\{\rho + (R-1)\left(r + \frac{1}{2}\frac{\kappa^2}{R}\right)\right\}, \quad \kappa = \sigma^{-1}(\mu - r). \tag{2.7b}$$

&

$$W_t = W_0 \exp\left\{R^{-1}\kappa B_t + (r + \frac{1}{R^2}\kappa^2(2R_1) - \gamma)t\right\}$$

&

Thrm 97. *The parameters in* (2.7), *above, are optimal for the Merton problem.*

[SEE NOTES FOR A PROOF]

MULTIPLE ASSETS:

[SAME AS BEFORE JUST A BIT MORE MESSY].

IF

$$dS_t^i = S_t^i \left\{ \sum_{j=1}^{N} \sigma^{ij} dB_t^j + \mu^i dt \right\}, \quad i = 1, \ldots, d$$

FOR

$$dW_t = r\left(W_t - \boldsymbol{n}_t \cdot \boldsymbol{S}_t\right) dt + \boldsymbol{n}_t \cdot d\boldsymbol{S}_t - c_t dt$$

THEN HJB

$$0 = \max_{\theta, c} \left\{ \underline{u(t,c)} + \partial_t V + \left(rw + \theta \cdot (\mu - r) - c\right) \partial_w V + \frac{1}{2} |\sigma^\top \theta|^2 \partial_{ww} V \right\}.$$

CRRA
AGAIN.

& OPTIMAL PARAMETERS ARE

$$\boldsymbol{\theta}^* = \frac{w}{R} (\sigma \sigma^\top)^{-1} (\boldsymbol{\mu} - \boldsymbol{r}), \quad c^* = \gamma^{-\frac{1}{R}} w$$

$$\gamma^{-\frac{1}{R}} = R^{-1} \left\{ \rho + (R-1)(r + \frac{1}{2} \frac{|\boldsymbol{\kappa}|^2}{R}) \right\} \qquad \boldsymbol{\kappa} = \sigma^{-1}(\boldsymbol{\mu} - \boldsymbol{r}).$$

GENERAL UTILITIES:

$$0 = \max_{\theta,c}\left\{\underbrace{u(t,c)}_{\substack{\text{INCREASING}\\ \text{CONCAVE}\\ \lim_{c\to\infty} u'(t,c)=0}} + \partial_t V + (rw + \theta\cdot(\mu - r) - c)\,\partial_w V + \frac{1}{2}|\sigma^\intercal\theta|^2\partial_{ww}V\right\}.$$

LET $u^*(t,z) = \max_c\{u(t,c) - zc\}$ & $J(t,z) = V(t,w) - wz$ FOR $z = \partial_w V(t,w)$.

Thrm 99. *The HJB equation can be written as*

$$0 = u^*(t,c) + \partial_t J - rz\partial_z J + \frac{1}{2}|\kappa|^2 z^2\partial_{zz}J$$

Moreover if we suppose that $u(t,x) = e^{-\rho t}u(x)$, for $u(x)$ concave and increasing, the HJB equation becomes

$$0 = u^*(y) - \rho j(y) + (\rho - r)yj'(y) + \frac{1}{2}|\kappa|^2 y^2 j''(y)$$

PROOF: FIRST WE SHOW $\partial_z J = -w$ ✓ , $\partial_{zz} J = -\frac{1}{\partial_{ww} V}$ ✓ $\partial_t J = \partial_t V$ ✓

$$J(z) = V(\underbrace{V'^{-1}}_{=w}(z)) - z\underbrace{V'^{-1}}_{=w}(z)$$

$$\therefore J'(z) = \cancel{\frac{d}{dz}(V'^{-1}(z)) \cdot V'(\underbrace{V'^{-1}(z)}_{z})} - \cancel{z\frac{d}{dz}V'^{-1}(z)} - \underbrace{V'^{-1}(z)}_{=w} = -w \quad ✓$$

&

$$\therefore J''(z) = -\frac{d}{dz}(V')^{-1}(z) = -\frac{1}{V''(V'^{-1}(z))} = -\frac{1}{V''(w)} \quad ✓$$

&

$$\frac{\partial J}{\partial t} = \partial_t V(t,w) + \cancel{\frac{dw}{dt}\partial_w V} - \cancel{z\frac{dw}{dt}} = \partial_t V \quad ✓$$

HJB IS

$$0 = \max_c \left\{ u(t,z) - c\partial_w V \right\} + \max_\theta \left\{ \theta\cdot(\mu - r)\partial_w V + \frac{1}{2}|\sigma\theta|^2 \partial_{ww} V \right\} + \partial_t V + rw\partial_w V$$

$$= u^*(t, \underbrace{\partial_w V}_{z}) - \frac{1}{2}|\kappa|^2 \underbrace{(\partial_w V)^2}_{z^2} \frac{1}{\underbrace{\partial_{ww} V}_{\partial_{zz} J}} + \underbrace{\partial_t V}_{\partial_t J} + r\underbrace{w}_{-\partial_z J}\underbrace{\partial_w V}_{z}$$

$$= u^*(t,z) + \frac{1}{2}|\kappa|^2 z^2 \partial_{zz} J + \partial_t J - rz\partial_z J.$$

PROOF (CONT)

[SIMILAR TO CRRA ARGUMENT]

CAN CHECK IF $u(t,x) = e^{-\rho t}u(x)$ THEN $V(t,w) = e^{-\rho t}V(w)$

LETTING $j(z) = V(w) - wz$, $z = \partial_w V(t,w)$

THEN, DIFFERENTIATING GIVES

$$J(t,z) = e^{-\rho t} j(y), \qquad \partial_t J = -\rho e^{-\rho t} j(y) + \rho e^{-\rho t} y j'(y)$$
$$\partial_z J = j'(y), \qquad \partial_{zz} J = e^{\rho t} j''(y)$$

SUBSTITUTING INTO HJB EQUATION ABOVE GIVES:

$$0 = \underbrace{u^*(t,z)}_{=e^{-\rho t}u^*(y)} + \underbrace{\partial_t J}_{=-\rho e^{-\rho t} j(y) + \rho e^{-\rho t} y j'(y)} - \underbrace{r z \partial_z J}_{r e^{-\rho t} y j'(y)} + \underbrace{\frac{1}{2}|\kappa|^2 z^2 \partial_{zz} J}_{\frac{1}{2}|\kappa|^2 y^2 e^{-\rho t} j''(y)}$$

$\square$

$$V(w) = \max_{(\boldsymbol{n}_t, c_t)_{t \geq 0} \in \mathcal{P}(w_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right].$$

$$j(z) = V(w) - wz$$