

10. DIFFUSION
CONTROL .

DIFFUSION CONTROL PROBLEM

For

$$dX_t = \mu_t(X_t, a_t)dt + \sigma_t(X_t, a_t) \cdot dB_t$$

MINIMIZE

$$L(x_0) := \underset{\substack{\Pi \in \mathcal{P} \\ \text{POLICIES}}}{\text{minimize}} C(x_0, \Pi) := \mathbb{E}_{x_0} \left[\int_0^T e^{-\alpha t} c_t(X_t, \pi_t) dt + e^{-\alpha T} c_T(X_T) \right] \quad (\text{DCP})$$



THE HJB EQUATION.

Def 67 (Hamilton-Jacobi-Bellman Equation). *For a Diffusion Control Problem (DCP), the equation*

$$0 = \min_{a \in \mathcal{A}} \left\{ c_t(x, a) + \partial_t L_t(x) + \mu_t(x, a) \cdot \partial_x L_t(x) + \frac{1}{2} [\sigma^T \sigma] \cdot \partial_{xx} L_t(x) - \alpha L_t(x) \right\}$$

(HJB)

INTUITIVE DEFINITION:

$$0 = \min_a \left\{ \text{"INSTANTANEOUS COST"} + \text{"DRIFT FOR ITO'S FORMULA"} \right\}$$

DERIVING HSB EQUATION

TAYLOR:

$$X_{t+\delta} - X_t = \mu_t(X_t, \pi_t)\delta + \sigma_t(X_t, \pi_t)(B_{T+\delta} - B_t)$$

COSTS:

$$C_t(x, \Pi) \approx \mathbb{E} \left[\sum_{t \in \{0, \delta, \dots, T-\delta\}} (1 - \alpha\delta)^{\frac{t}{\delta}} c_t(X_t, \pi_t)\delta + (1 - \alpha\delta)^{\frac{T}{\delta}} c_T(X_T) \right].$$

NOW IS AN MDP.

BELLMAN EQN IS

$$\underbrace{L_t(x) = \min_{a \in \mathcal{A}} \{c_t(x, a)\delta + (1 - \alpha\delta)\mathbb{E}_{x,a} [L_{t+\delta}(X_{t+\delta})]\}}_{\text{THUS}}$$

$$0 = \min_{a \in \mathcal{A}} \left\{ c_t(x, a) + \frac{1}{\delta} \mathbb{E}_{x,a} [\underbrace{L_{t+\delta}(X_{t+\delta}) - L_t(x)}_{\text{EXPAND.}}] - \alpha \mathbb{E}_{x,a} [L_{t+\delta}(X_{t+\delta})] \right\}.$$

ITÔ'S FORMULA

$$\begin{aligned} & L_{t+\delta}(X_{t+\delta}) - L_t(X_t) \\ & \approx \left[\partial_t L + \mu_t(X_t, \pi_t) \cdot \partial_x L + \frac{\sigma_t(X_t, \pi_t)^2}{2} \partial_{xx} L \right] \delta + \partial_x L \cdot \sigma_t(X_t, \pi_t) \cdot (B_{t+\delta} - B_t) \end{aligned}$$

THUS

$$\frac{1}{\delta} \mathbb{E}_{x,a} [L_{t+\delta}(X_{t+\delta}) - L_t(x)] = \partial_t L + \mu_t(X_t, \pi_t) \cdot \partial_x L + \frac{\sigma_t(X_t, \pi_t)^2}{2} \partial_{xx} L$$

SUBSTITUTING BACK GIVES:

$$0 = \min_{a \in \mathcal{A}} \left\{ c_t(x, a) + \partial_t L + \mu_t(x, a) \cdot \partial_x L + \frac{\sigma_t(x, a)^2}{2} \partial_{xx} L - \alpha L_t(x) \right\}.$$

AKA. THE HJB EQN.

A MARTINGALE PRINCIPLE:

Thrm 68 (Davis-Varaiya Martingale Principle of Optimality). Suppose that there exists a function $L_t(x)$ with $L_T(x) = e^{-\alpha T} c_T(x)$ and such that for any policy Π with states X_t

$$M_t = L_t(X_t) + \int_0^t e^{-\alpha \tau} c_\tau(X_\tau, \Pi) d\tau$$

is a sub-martingale and, moreover that for some policy Π^* , M_t is a martingale then Π^* is optimal and

$$L_0(X_0) = \min_{\Pi \in \mathcal{P}} \mathbb{E} \left[\int_0^T e^{-\alpha \tau} c_\tau(X_\tau, \pi_\tau) d\tau + c_T(X_T) \right].$$

PROOF:

$$L_0(x) = M_0 \leq \mathbb{E}_x[M_T] = \mathbb{E}_x \left[\int_0^T e^{-\alpha s} c_s(X_s, \pi_s) ds + \underbrace{L_T(X_T)}_{e^{-\alpha T} c_T(X_T)} \right]$$

$$\therefore L_0(x_0) \leq C(x, \pi) \quad \forall \pi$$

& IF M_y FOR π^* THEN

$$L_0(x_0) = C(x, \pi^*) \quad \text{FOR } \pi^*$$

$\therefore \pi^*$ IS OPTIMAL

B.

