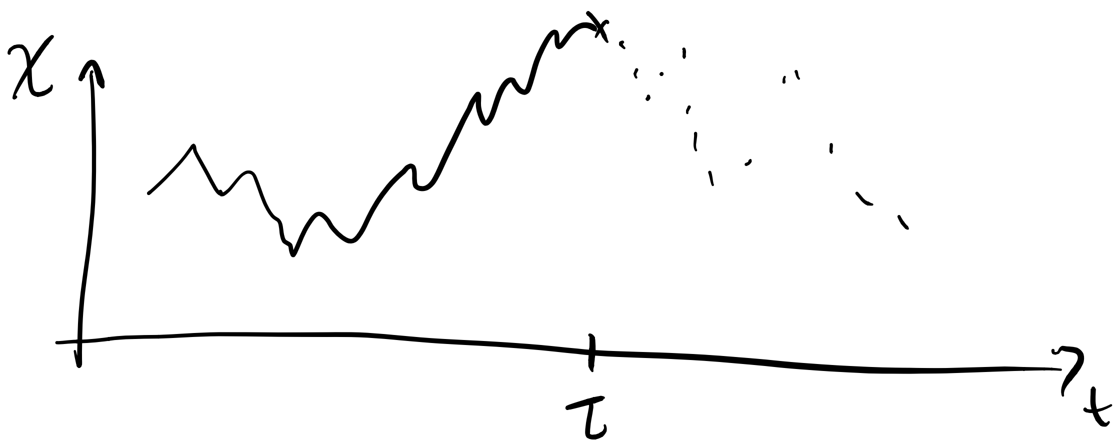
6. OPTIMAL STOPPING

DEFINITION:

- An Optimal Stopping Problem is an Markov Decision Process where there are two actions: $a = 0$ meaning to stop, and $a = 1$ meaning to continue. Here there are two types of costs

$$c(x,a) = \begin{cases} \kappa(x), & \text{for } a = 0 \quad \textit{(the stopping cost)} \\ c(x), & \text{for } a = 1 \quad \textit{(the continuation cost)}, \end{cases}$$

This defines a *stopping problem*.



OBJECTIVE:

$$C(x) = \min_{\tau} \mathbb{E}\left[\sum_{t=0}^{\tau-1} c(x_t) + k(x_\tau) \right]$$

BELLMAN EQUATION:

Assuming that time is finite, the Bellman equation is

$$C_s(x) = \min \left\{ k(x), c(x) + \mathbb{E}_x[C_{s-1}(\hat{X})] \right\}$$

for $s \in \mathbb{N}$ and $C_0(x) = k(x)$.

OSLA & CLOSED SETS:

Def 44 (OLSA rule). *In the one step lookahead (OSLA) rule we stop when ever $x \in S$ where*

$$S = \{x : \underbrace{k(x)}_{\text{STOP NOW}} \leq \underbrace{c(x) + \mathbb{E}_x[k(\hat{X})]}_{\text{STOP NEXT}}\}.$$

We call S the stopping set. In words, you stop whenever it is better stop now rather than continue one step further and then stop.

Def 45 (Closed Stopping Set). *We say the set $S \subset \mathcal{X}$ is closed, it once inside that said you cannot leave, i.e.*

$$P_{xy} = 0, \qquad \forall x \in S, y \notin S.$$

Prop 46. *If, for the finite time stopping problem, the set S given by the one step lookahead rule is closed then the one step lookahead rule is an optimal policy.*

[i.e. ONE STEP LEFT]

PROOF: BY INDUCTION, IF $\underset{d}{s}=1$ THEN OSLA IS OPTIMAL ✓

SUPPOSE OSLA IS OPTIMAL FOR s, [WE SHOW ITS TRUE FOR $s+1$].

IF $x \notin \mathcal{X}$, CLEARLY IS OPTIMAL TO CONTINUE [AS CONTINUING 1 MORE STEP & STOPPING IS BETTER!]

IF $x \in \mathcal{X}$, THEN

$$L_{s+1}(x) = \min\{ k(x), c(x) + \mathbb{E}_x[\underbrace{C_s(\hat{x})}] \}$$

$\hat{x} \in S$ SINCE S IS CLOSED

& SINCE OSLA IS OPTIMAL AT s STEPS

& $\hat{x} \in S$ OPTIMAL TO STOP

$\therefore \; C_s(\hat{x}) = k(\hat{x})$

$$= \min\{k(x), c(x) + \mathbb{E}_x[K(\hat{x})]\} = K(x)$$

$\therefore$ OPTIMAL TO STOP IF $x \in S$

SINCE $x \in S$

$\square$.

OPTIMAL STOPPING IN INFINITE TIME:

Prop 47. *If the following two conditions hold*

- $K = \max_x k(x) < \infty$, $\min_x k(x) \geq 0$
- $C = \min_x c(x) > 0$
- S IS CLOSED.

then the One-Step-Lookahead-Rule is optimal.

PROOF: NOTICE IF WE CAN SHOW $L_s(x) \to L_\infty(x)$,

AS $L_s(x) = K(x)\ \forall s\ \forall x \in S$, THEN $L_\infty(x) = K(x)\ \forall x \in S$

$\therefore$ IS OPTIMAL TO STOP FOR $x \in S$.

τ^* OPTIMAL STOPPING TIME.

$$\underbrace{L_\infty(x)}_{\text{STOP AT ANY TIME}} \leq \underbrace{L_s(x)}_{\text{MUST STOP BEFORE } s} \leq \underbrace{L_\infty(x) + K\,\mathbb{P}(\tau^* \geq s)}_{\text{FOLLOW OPTIMAL POLICY FOR INFINITE TIME THEN STOP AT } s.}$$

NOW LETS BOUND $\mathbb{P}_x(\tau^* > s)$:

$$\mathbb{P}_x(\tau^* \geq s) \leq \frac{1}{sC} \mathbb{E}_x\left[\underbrace{\left\{ \sum_{t=0}^{\tau^*-1} c(x_t) + K(x_{\tau^*}) \right\}}_{\text{THIS IS BIGGER THAN } sC} \mathbb{I}[\tau^* \geq s] \right]$$

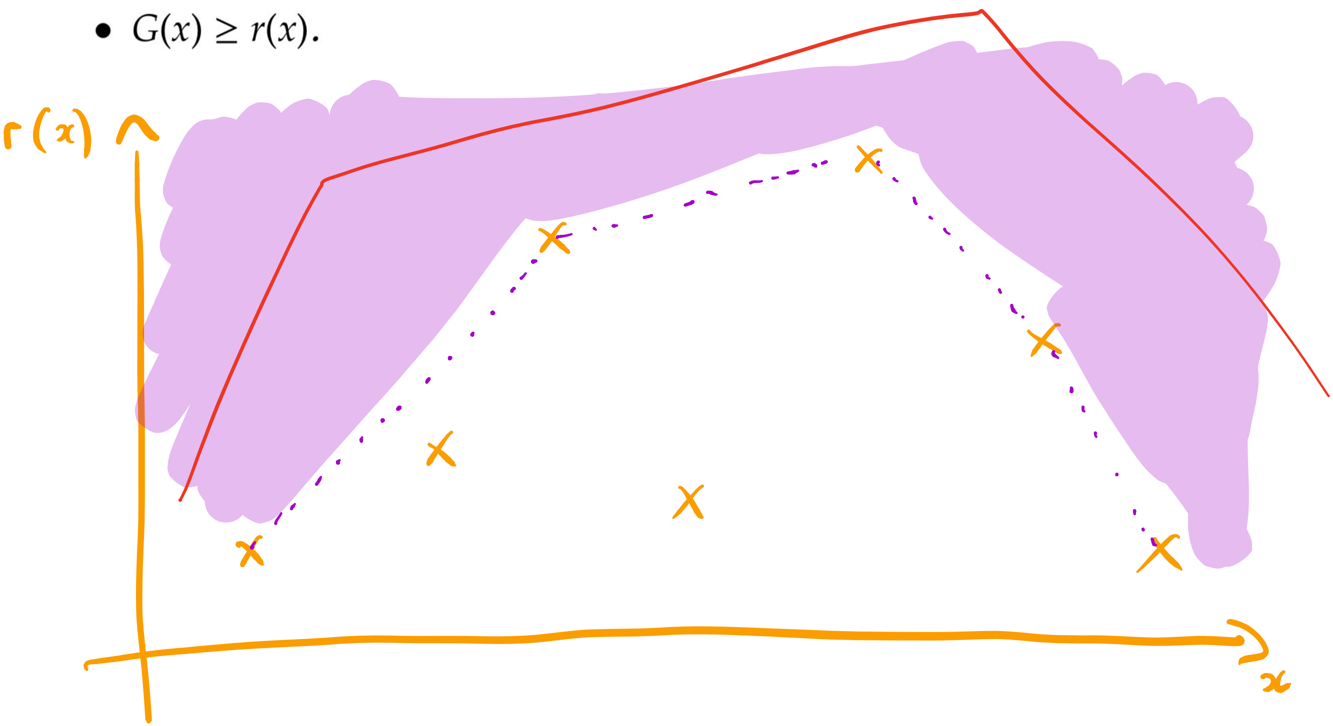
$$\leq \frac{1}{sC} L_\infty(x) \leq \frac{K(x)}{sC} \leq \frac{K}{sC}$$

$\therefore \mathbb{P}_x(\tau^* \geq s) \xrightarrow[s\to\infty]{} 0 \quad \therefore L_s(x) \to L_\infty(x) \quad \therefore$ OSLA IS OPTIMAL $\square$.

STOPPING A RANDOM WALK:

Def 48 (Concave Majorant)**.** *For a function $r : \{0, \ldots, N\} \to \mathbb{R}_+$ a concave majorant is a function G such that*

- $G(x) \geq \frac{1}{2}G(x-1) + \frac{1}{2}G(x+1)$
- $G(x) \geq r(x)$.



Prop 49 (Stopping a Random Walk). *Let X_t be a symmetric random walk on $\{0, \dots, N\}$ where the process is automatically stopped at 0 and N. For each $x \in \{0, \dots, N\}$, there is a positive reward of $r(x)$ for stopping. We are asked to maximize*

$$\mathbb{E}[r(X_T)]$$

where T is our chosen stopping time. The optimal value function $V(x)$ is the minimal concave majorant, and that it is optimal to stop whenever $V(x) = r(x)$.

PROOF:

BELLMAN EQUATION IS

$$V(x) = \max\left\{ r(x) \, , \, \frac{1}{2} V(x-1) + \frac{1}{2} V(x+1) \right\}$$

$\therefore$ V IS A CONCAVE MAJORANT.

PROOF (CONT): SUPPOSE $G(x)$ IS A CONCAVE MAJORANT.

LET $V_s(x)$ - BE s-STEP OPTIMAL SOLUTION

$$V_0(x) = r(x) \quad \therefore \quad V_0(x) \le G(x)$$

SUPPOSE [BY INDUCTION] $V_s(x) \le G(x)$

THEN

$$\begin{aligned} V_{s+1}(x) &= \max\left\{ r(x),\ \frac{V_s(x-1)}{2} + \frac{V_s(x+1)}{2} \right\} \\ &\le \max\left\{ r(x),\ \frac{G(x-1)}{2} + \frac{G(x+1)}{2} \right\} \\ &\le G(x) \qquad \therefore \quad V_s(x) \le G(x) \ \forall s \end{aligned}$$

AS VALUE ITERATION CONVERGES $V(x) = \lim_{s\to\infty} V_s(x) \le G(x)$

$\therefore$ IS MINIMAL CONCAVE MAJORANT. $\square$